

BERTINI THEOREMS FOR SMOOTH HYPERSURFACE SECTIONS CONTAINING A SUBSCHEME OVER FINITE FIELDS

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ABSTRACT. We show the existence of a hypersurface that contains a given closed subscheme of a projective space over a finite field and intersects a smooth quasi-projective scheme smoothly, under some condition on the dimension. This generalizes a Bertini theorem by Poonen and is the finite field analogue of a Bertini theorem by Altman and Kleiman. Furthermore, we add the possibility of modifying finitely many local conditions of the hypersurface. We show that the condition on the dimension is fulfilled for schemes with simple normal crossings and give an application to embeddings into smooth schemes.

1. INTRODUCTION

For a smooth subscheme X of the projective space \mathbb{P}_k^n over a finite field k , Poonen proved the existence of smooth hypersurface sections using a geometric closed point sieve ([Poo04]). With this sieve method, he also proved in [Poo08] that the hypersurface can be assumed to contain a given closed subscheme $Z \subseteq \mathbb{P}_k^n$, provided that $Z \cap X$ is smooth and $\dim X > 2 \dim Z \cap X$. This was already known for infinite fields ([Blo71]); in this case, [KA79] also showed an analogue where the intersection $Z \cap X$ is not smooth, assuming $Z \subseteq X$ and another condition on the dimension. In this paper, we prove this analogue over finite fields as a special case of a result where we add the possibility to prescribe finitely many local conditions on the hypersurface. We show that the condition on the dimension is fulfilled when $Z \cap X$ is an equidimensional scheme with simple normal crossings and prove embedding results for schemes over finite fields.

Let \mathbb{F}_q be a finite field of $q = p^a$ elements. Let $S = \mathbb{F}_q[x_0, \dots, x_n]$ be the homogeneous coordinate ring of the projective space \mathbb{P}^n over \mathbb{F}_q and let $S_d \subseteq S$ be the \mathbb{F}_q -subspace of homogeneous polynomials of degree d . Let $S_{\text{homog}} = \bigcup_{d \geq 0} S_d$ and let S'_d be the set of all polynomials in $\mathbb{F}_q[x_0, \dots, x_n]$ of degree $\leq d$.

For a scheme X of finite type over \mathbb{F}_q , we define the zeta function as

$$\zeta_X(s) := \prod_{P \in X \text{ closed}} (1 - q^{-s \deg P})^{-1}.$$

This product converges for $\text{Re}(s) > \dim X$.

Let Z be a fixed closed subscheme of \mathbb{P}^n . For $d \in \mathbb{Z}_{\geq 0}$ let I_d be the \mathbb{F}_q -subspace of polynomials $f \in S_d$ vanishing on Z , and $I_{\text{homog}} = \bigcup_{d \geq 0} I_d$. For a polynomial $f \in I_d$ let $H_f = \text{Proj}(S/(f))$ be the hypersurface defined by f . As in Poonen's paper ([Poo08]), we define the density of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ by

$$\mu_Z(\mathcal{P}) := \lim_{d \rightarrow \infty} \frac{\#(\mathcal{P} \cap I_d)}{\#I_d},$$

if the limit exists. We have to use this density relative to I_{homog} and cannot measure the density using the definition of [Poo04], since if the dimension of Z is positive,

the density of I_{homog} would always be zero (cf. Lemma 3.1 [CP13]). We further define the upper and lower density $\overline{\mu}_Z(\mathcal{P})$ and $\underline{\mu}_Z(\mathcal{P})$ of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ by

$$\overline{\mu}_Z(\mathcal{P}) := \limsup_{d \rightarrow \infty} \frac{\#(\mathcal{P} \cap I_d)}{\#I_d}, \quad \text{and} \quad \underline{\mu}_Z(\mathcal{P}) = \liminf_{d \rightarrow \infty} \frac{\#(\mathcal{P} \cap I_d)}{\#I_d}.$$

We define the embedding dimension of X at x to be $e(x) = \dim_{\kappa(x)}(\Omega_{X|\mathbb{F}_q}^1(x))$. Let

$$X_e = X(\Omega_{X|\mathbb{F}_q}^1, e)$$

be the subscheme such that a scheme morphism $f : T \rightarrow X$ factors through X_e if and only if $f^*\Omega_{X|\mathbb{F}_q}^1$ is locally free of rank e . Then X_e is the locally closed subscheme of X where the embedding dimension of X is e .

Following Poonen ([Poo04] Lemma 1.2), we may impose local conditions on the hypersurface at a finite subscheme Y of \mathbb{P}^n . For a polynomial $f \in I_d$ we define $f|_Y \in H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$ as follows: on each connected component Y_i of Y let $f|_{Y_i}$ be equal to the restriction of $x_j^{-d}f$ to Y_i , where $j = j(i)$ is the smallest $j \in \{0, 1, \dots, n\}$ such that the coordinate x_j is invertible on Y_i .

Theorem 1.1. *Let X be a quasi-projective subscheme of \mathbb{P}^n over \mathbb{F}_q and let Z be a closed subscheme of \mathbb{P}^n . Let Y be a finite subscheme of \mathbb{P}^n , such that $U := X - (X \cap Y)$ is smooth of dimension $m \geq 0$ and $Y \cap Z = \emptyset$. Let $V = Z \cap U$ be the intersection and let T be a subset of $H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$. If*

$$\mathcal{P} = \{f \in I_{\text{homog}} : H_f \cap U \text{ is smooth of dimension } m-1 \text{ and } f|_Y \in T\},$$

then the following holds:

(1) *If $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$, then*

$$\mu_Z(\mathcal{P}) = \frac{\#T}{\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)} \frac{\zeta_V(m+1)}{\zeta_U(m+1) \prod_{e=0}^{m-1} \zeta_{V_e}(m-e)} \neq 0.$$

(2) *If $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} \geq m$ or $V_m = \emptyset$, then $\mu_Z(\mathcal{P}) = 0$.*

The special case for $Y = \emptyset$ and $T = \{0\}$ yields the following result of [Wut14] (Theorem 2.1), which has meanwhile also been proved independently by Gunther ([Gun15] Theorem 1.1):

Theorem 1.2. *Let X be a quasi-projective subscheme of \mathbb{P}^n which is smooth of dimension $m \geq 0$ over \mathbb{F}_q . Let Z be a closed subscheme of \mathbb{P}^n and let $V := Z \cap X$ be the intersection. If*

$$\mathcal{P} = \{f \in I_{\text{homog}} : H_f \cap X \text{ is smooth of dimension } m-1\},$$

then the following holds:

(1) *If $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$, then*

$$\mu_Z(\mathcal{P}) = \frac{\zeta_V(m+1)}{\zeta_X(m+1) \prod_{e=0}^{m-1} \zeta_{V_e}(m-e)} = \frac{1}{\zeta_{X-V}(m+1) \prod_{e=0}^{m-1} \zeta_{V_e}(m-e)} \neq 0.$$

In particular, there exists a hypersurface H of degree $d \gg 1$ containing Z such that $H \cap X$ is smooth of dimension $m-1$.

(2) *If $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} \geq m$ or $V_m \neq \emptyset$, then $\mu_Z(\mathcal{P}) = 0$.*

Remark 1.3. (i) If we choose Z to be empty, then the conditions of Theorem 1.2(i) are fulfilled and Theorem 1.2 gives Theorem 1.1 of [Poo04].

- (ii) If the intersection $V = Z \cap X$ is smooth of dimension $l \geq 0$ as required in [Poo08], then the condition on the dimension in Theorem 1.2 implies $l + \dim V = 2l < m$ and therefore Theorem 1.2 (i) also yields the statement of Theorem 1.1 of [Poo08].
- (iii) The density in Theorem 1.2 is independent of the embedding $X \hookrightarrow \mathbb{P}^n$.
- (iv) Note that the density is relative to I_{homog} and does not depend on points outside of Z ; therefore we must fix Z at the beginning and cannot, in general, compare two densities obtained for different closed schemes.

Corollary 1.4. *Let X be a quasi-projective subscheme of \mathbb{P}^n that is smooth of dimension $m \geq 0$ over \mathbb{F}_q at all but finitely many closed points P_1, \dots, P_r . Let Z be a closed subscheme of \mathbb{P}^n that does not contain any of those points and let $V = Z \cap X$ be the intersection. Suppose $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$. Then for $d \gg 1$, there exists a hypersurface H of degree d that contains Z but none of the points P_1, \dots, P_r , such that $H \cap X$ is smooth of dimension $m - 1$.*

Proof. Let $Y_i = \text{Spec } \kappa(P_i)$ and $Y = \bigcup_{i=1}^r Y_i$. Then $U = X - (X \cap Y)$ is smooth of dimension $m \geq 0$, $Y \cap Z = \emptyset$ and $H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) = \prod_{i=1}^r \mathcal{I}_{Z, P_i} \cdot \kappa(P_i)$. We define $T \subseteq H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$ to be the nonempty set of elements that are nonzero in every component of the above product. For $f|_Y \in T$ this implies $P_i \notin H_f$ for all $1 \leq i \leq r$, and thus $H_f \cap Y = \emptyset$.

Applying Theorem 1.1, we get the existence of a hypersurface H of degree $d \gg 1$ that does not intersect Y and therefore contains none of the points P_1, \dots, P_r ; further it intersects U and thus also X smoothly. \square

Let V be a subscheme of \mathbb{P}^n and let W_1, \dots, W_s be the irreducible components of V . We say that V has simple normal crossings if W_i is smooth for any i , $\bigcap_{i \in I} W_i$ is smooth and $\text{codim}_V \bigcap_{i \in I} W_i = \#I - 1$ for any subset $I \subseteq \{1, \dots, s\}$.

Corollary 1.5. *Let the notations be as in Theorem 1.2. Suppose V is equidimensional of dimension l and has simple normal crossings. If furthermore $2l < m$ holds, then there exists a hypersurface H containing Z such that $H \cap X$ is smooth of dimension $m - 1$.*

Proof. We show that the conditions of Theorem 1.2 (i) are fulfilled for the schemes $V_e = V_{l+k}$ of the flattening stratification of V for $0 \leq k \leq m - l$. One can prove by induction that V_{l+k} is contained in the union of all intersections of $k + 1$ irreducible components of V ; more precisely, if a point P is in the intersection of exactly k irreducible components of V , then $e_V(P) \leq l + k - 1$. This yields $\dim V_{l+k} + l + k \leq 2l$ for $0 \leq l \leq m$. Hence, if $2l < m$ holds, then the conditions of Theorem 1.2 are satisfied and the corollary follows.

Note that V_m is empty, since V_{m-1} is contained in the union of the intersections of $m - l$ irreducible components, and this union is already of dimension zero. As V has simple normal crossings, the intersection of $m - l + 1$ components, which contains V_m , must be empty. \square

Corollary 1.6. *Let Z be a quasi-projective scheme over \mathbb{F}_q satisfying*

$$\max\{e + \dim Z_e\} \leq r.$$

Then Z can be embedded in a smooth scheme Y over \mathbb{F}_q of dimension r . If Z is projective, we can choose Y to be projective.

In particular, if Z is of dimension l with simple normal crossings, then there exists a smooth $2l$ -dimensional scheme Y over \mathbb{F}_q in which Z can be embedded. Y can be chosen projective if Z is projective.

Proof. (cf [KA79] Theorem 8) If Z is projective, let Z be closed in $X = \mathbb{P}^n$. (For the quasi-projective case, embed Z in some open smooth subscheme $X \subseteq \mathbb{P}^N$ of dimension n .) By assumption, we have $\max\{e + \dim Z_e\} \leq r$ and Theorem 1.2 gives a hypersurface H containing Z which is smooth of dimension $n - 1$. Inductively, we get a smooth scheme Y of dimension r containing Z . The second part follows, since $\max\{e + \dim Z_e\} \leq 2l$, as seen in the proof of Corollary 1.5. \square

The proof of Theorem 1.1 uses the closed point sieve introduced in [Poo04] and is parallel to the one in [Poo08]. Gunther also used the sieve proof to show Theorem 1.2.

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2. SINGULAR POINTS OF LOW DEGREE

Let $\mathcal{I}_Z \subseteq \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of Z ; then $I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d))$. As in [Poo08], we fix an integer c such that $S_1 I_d = I_{d+1}$ for all $d \geq c$.

Lemma 2.1. ([Poo08], Lemma 2.1.) *Let Y be a finite subscheme of \mathbb{P}^n over \mathbb{F}_q . Let*

$$\phi_d : I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \rightarrow H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$$

be the map induced by the map of sheaves $\mathcal{I}_Z \rightarrow \mathcal{I}_Z \cdot \mathcal{O}_Y$ on \mathbb{P}^n . Then ϕ_d is surjective for $d \geq c + \dim H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$.

Remark 2.2. In the situation of Theorem 1.1, for a closed point $P \in V$, we have $e_V(P) = \dim_{\kappa(P)} \mathfrak{m}_{U,P} / (\mathcal{I}_{Z,P}, \mathfrak{m}_{U,P}^2)$; in particular, $\dim U \geq e_V(P)$.

Lemma 2.3. *Let $\mathfrak{m} \subseteq \mathcal{O}_U$ be the ideal sheaf of a closed point $P \in U$. Let $C \subseteq U$ be the closed subscheme of \mathbb{P}^n corresponding to the ideal sheaf $\mathfrak{m}^2 \subseteq \mathcal{O}_U$. Then for all $d \in \mathbb{Z}_{\geq 0}$,*

$$\#H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C(d)) = \begin{cases} q^{(m+1) \deg P}, & \text{if } P \notin V, \\ q^{(m-e_V(P)) \deg P}, & \text{if } P \in V. \end{cases}$$

Proof. Since C is a finite scheme, we may ignore the twist, i.e. $H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C(d)) = H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C)$. Taking cohomology of $0 \rightarrow \mathcal{I}_Z \cdot \mathcal{O}_C \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{Z \cap C} \rightarrow 0$ on the 0-dimensional scheme C and using [Har93] Theorem III 2.7 yields an exact sequence

$$0 \rightarrow H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C) \rightarrow H^0(C, \mathcal{O}_C) \rightarrow H^0(C, \mathcal{O}_{Z \cap C}) \rightarrow 0.$$

There is a filtration of $H^0(C, \mathcal{O}_C) = \mathcal{O}_{U,P} / \mathfrak{m}_{U,P}^2$ whose quotients are vector spaces of dimensions m and 1 respectively over the residue field $\kappa(P)$ of P . Thus, $\#H^0(C, \mathcal{O}_C) = \#\kappa(P)^{m+1} = q^{(m+1) \deg P}$. Next we determine $\#H^0(C, \mathcal{O}_{Z \cap C})$. If $P \in U - V$, then $H^0(C, \mathcal{O}_{Z \cap C}) = 0$. If $P \in V$, then $H^0(C, \mathcal{O}_{Z \cap C})$ has a filtration whose quotients have dimensions 1 and $e_V(P)$ over $\kappa(P)$ by Remark 2.2. Hence,

$$\#H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C) = \begin{cases} q^{(m+1) \deg P}, & \text{if } P \notin V, \\ q^{(m+1) \deg P} / q^{(e_V(P)+1) \deg P}, & \text{if } P \in V. \end{cases}$$

\square

For a scheme U of finite type over \mathbb{F}_q we define $U_{<r}$ to be the set of closed points of U of degree $< r$. Let $U_{>r}$ be defined similarly.

Lemma 2.4 (Singularities of low degree). *Define*

$$\mathcal{P}_r := \{f \in I_{\text{homog}} : H_f \cap U \text{ is smooth of dimension } m-1 \\ \text{at all points } P \in U_{<r} \text{ and } f|_Y \in T\}.$$

Then

$$\mu_Z(\mathcal{P}_r) = \frac{\#T}{\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)} \prod_{P \in (U-V)_{<r}} (1 - q^{-(m+1)\deg P}) \prod_{e=0}^m \prod_{P \in (V_e)_{<r}} (1 - q^{-(m-e)\deg P}).$$

Proof. Let $U_{<r} = \{P_1, \dots, P_s\}$. Let \mathfrak{m}_i be the ideal sheaf of P_i on U and let C_i be the closed subscheme of U corresponding to the ideal sheaf $\mathfrak{m}_i^2 \subseteq \mathcal{O}_U$. Then $H_f \cap U$ is not smooth of dimension $m-1$ at P_i if and only if the restriction of f to a section of $\mathcal{I}_Z \cdot \mathcal{O}_{C_i}(d)$ is equal to zero. Since we also want $f|_Y$ to be in T , the set $\mathcal{P}_r \cap I_d$ is the inverse image of

$$T \times \prod_{i=1}^s (H^0(C_i, \mathcal{I}_Z \cdot \mathcal{O}_{C_i}) \setminus \{0\})$$

under the \mathbb{F}_q -linear composition

$$\begin{aligned} \phi_d : I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) &\rightarrow H^0(Y \cup C, \mathcal{I}_Z \cdot \mathcal{O}_{Y \cup C}(d)) \\ &\cong H^0(Y \cup C, \mathcal{I}_Z \cdot \mathcal{O}_{Y \cup C}) \cong H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) \times \prod_{i=1}^s H^0(C_i, \mathcal{I}_Z \cdot \mathcal{O}_{C_i}), \end{aligned}$$

where $C := \bigcup_{i=1}^s C_i$. The first isomorphism is the untwisting by multiplication by x_j^{-d} component by component as in the definition of $f|_Z$. Note that at this point, we need the restriction $Y \cap Z = \emptyset$. For d large enough, the map ϕ_d is surjective and it follows that

$$\mu_Z(\mathcal{P}_r) = \frac{\#T}{\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)} \frac{\# \prod_{i=1}^s (H^0(C_i, \mathcal{I}_Z \cdot \mathcal{O}_{C_i}) \setminus \{0\})}{\# \prod_{i=1}^s H^0(C_i, \mathcal{I}_Z \cdot \mathcal{O}_{C_i})}.$$

Applying Lemma 2.3 yields the result. \square

Corollary 2.5. *Let $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$, then*

$$\lim_{r \rightarrow \infty} \mu_Z(\mathcal{P}_r) = \frac{\#T}{\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)} \frac{\zeta_V(m+1)}{\zeta_U(m+1) \prod_{e=0}^{m-1} \zeta_{V_e}(m-e)}.$$

Proof. The first product $\prod_{P \in (U-V)_{<r}} (1 - q^{-(m+1)\deg P})$ in Lemma 2.4 converges anyway, since $m+1 > \dim(U-V)$. For all $0 \leq e \leq m-1$, the product $\prod_{P \in (V_e)_{<r}} (1 - q^{-(m-e)\deg P})$ is just the partial product used in the definition of the zeta function of V_e . This converges for $m-e > \dim V_e$, i.e. for $\dim V_e + e < m$. \square

Proof of Theorem 1.1 (ii). The inclusion $\mathcal{P} \subseteq \mathcal{P}_r$ implies $\mu_Z(\mathcal{P}) \leq \mu_Z(\mathcal{P}_r)$, and thus it suffices to show that $\mu_Z(\mathcal{P}_r) = 0$. If $e + \dim V_e < m$ fails for some e , then the corresponding product in Lemma 2.4, which is the inverse of the partial product defining the zeta function of V_e , tends to zero as the zeta function has a pole at $\dim V_e$ (cf. [Tat65] §4). If $V_m \neq \emptyset$, the factor $(1 - q^{-(m-m)\deg P})$ appearing in the density of \mathcal{P}_r in Lemma 2.4 is equal to zero; hence $\mu_Z(\mathcal{P}_r)$ is zero. \square

3. SINGULAR POINTS OF MEDIUM DEGREE

Lemma 3.1. *Let $P \in U$ be a closed point of degree $\leq \frac{d-c}{m+1}$. Then the fraction of polynomials $f \in I_d$ such that $H_f \cap U$ is not smooth of dimension $m-1$ at P is equal to*

$$\begin{cases} q^{-(m+1)\deg P}, & \text{if } P \notin V, \\ q^{-(m-e_V(P))\deg P}, & \text{if } P \in V. \end{cases}$$

Proof. Let $C \subseteq U$ be defined as in Lemma 2.3. Then $H_f \cap U$ is not smooth of dimension $m-1$ at P if and only if the restriction of f to a section of $\mathcal{I}_Z \cdot \mathcal{O}_C(d)$ is equal to zero. As we have an isomorphism $H^0(\mathbb{P}^n, \mathcal{I}_Z(d))/\ker \phi_d \cong H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C(d))$ by Lemma 2.1, Lemma 2.3 applied to C gives the fractions. \square

Lemma 3.2 (Singularities of medium degree). *Let*

$$\mathcal{Q}_r^{\text{medium}} := \bigcup_{d \geq 0} \{f \in I_d : \text{there exists a point } P \in U \text{ with } r \leq \deg P \leq \frac{d-c}{m+1} \text{ such that } H_f \cap U \text{ is not smooth of dimension } m-1 \text{ at } P\}.$$

Then $\lim_{r \rightarrow \infty} \bar{\mu}_Z(\mathcal{Q}_r^{\text{medium}}) = 0$.

Proof. Since the number of points P of degree g in U is at most $\#U(\mathbb{F}_{q^g})$, Lemma 3.1 yields

$$\frac{\#(\mathcal{Q}_r^{\text{medium}} \cap I_d)}{\#I_d} \leq \sum_{e=0}^m \sum_{g=r}^{\infty} \#V_e(\mathbb{F}_{q^g}) q^{-(m-e)g} + \sum_{g=r}^{\infty} \#(U-V)(\mathbb{F}_{q^g}) q^{-(m+1)g}.$$

By ([LW54] Lemma 1), there exist constants a_e and a for V_e and $U-V$ that depend only on V_e and $U-V$, respectively, such that $\#V_e(\mathbb{F}_{q^g}) \leq a_e q^{g \dim V_e}$ and $\#(U-V)(\mathbb{F}_{q^g}) \leq a q^{g \dim(U-V)}$. Using the assumptions $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$, we obtain

$$\frac{\#(\mathcal{Q}_r^{\text{medium}} \cap I_d)}{\#I_d} \leq O(q^{-r}),$$

which tends to zero for $r \rightarrow \infty$. \square

4. SINGULAR POINTS OF HIGH DEGREE

Lemma 4.1 (Singularities of high degree off V). *Define*

$$\mathcal{Q}_{U-V}^{\text{high}} := \bigcup_{d \geq 0} \{f \in I_d : \text{there exists a point } P \in (U-V)_{> \frac{d-c}{m+1}} \text{ such that } H_f \cap U \text{ is not smooth of dimension } m-1 \text{ at } P\}.$$

Then $\bar{\mu}_Z(\mathcal{Q}_{U-V}^{\text{high}}) = 0$.

Proof. This is the statement of Lemma 4.2. in [Poo08]; the proof does not use the fact that V is smooth. \square

Lemma 4.2 (Singularities of high degree on V). *Define*

$$\mathcal{Q}_V^{\text{high}} := \bigcup_{d \geq 0} \{f \in I_d : \text{there exists a point } P \in V_{> \frac{d-c}{m+1}} \text{ such that } H_f \cap U \text{ is not smooth of dimension } m-1 \text{ at } P\}.$$

Then $\bar{\mu}_Z(\mathcal{Q}_V^{\text{high}}) = 0$.

Proof. If the lemma is proven for all subsets U_i of a finite open cover of U , then it holds for U as well. Hence we can assume without loss of generality, that $U \subseteq \mathbb{A}_{\mathbb{F}_q}^n = \{x_0 \neq 0\} \subseteq \mathbb{P}_{\mathbb{F}_q}^n$ is affine. We identify S_d with the space of polynomials $S'_d \subseteq \mathbb{F}_q[x_1, \dots, x_n] = A$ of degree $\leq d$ by setting $x_0 = 1$. This dehomogenization also identifies I_d with a subspace $I'_d \subseteq S'_d$.

Let P be a closed point of U . Since U is smooth, we can choose a system of local parameters $t_1, \dots, t_n \in A$ on \mathbb{A}^n such that $t_{m+1} = \dots = t_n = 0$ defines U locally at P . Then dt_1, \dots, dt_n are a basis for the stalk of $\Omega_{\mathbb{A}^n | \mathbb{F}_q}^1$ at P and dt_1, \dots, dt_m are a basis for the stalk of $\Omega_{U | \mathbb{F}_q}^1$ at P . We will show that the probability that $H_f \cap U$ is not smooth at a point in V_e tends to zero for $d \rightarrow \infty$ and any e .

Consider the map $\Omega_{U | \mathbb{F}_q}^1 \otimes \mathcal{O}_V \rightarrow \Omega_{V | \mathbb{F}_q}^1$, which is surjective ([Har93] Proposition II.8.12). Tensoring it with \mathcal{O}_{V_e} gives a surjective map $\phi : \Omega_{U | \mathbb{F}_q}^1 \otimes \mathcal{O}_{V_e} \rightarrow \Omega_{V | \mathbb{F}_q}^1 \otimes \mathcal{O}_{V_e}$, where $\Omega_{U | \mathbb{F}_q}^1 \otimes \mathcal{O}_{V_e}$ and $\Omega_{V | \mathbb{F}_q}^1 \otimes \mathcal{O}_{V_e}$ are locally free sheaves of rank m and e , resp. Hence, dt_1, \dots, dt_{m-e} form a basis of the kernel of ϕ at P and dt_{m-e+1}, \dots, dt_m a basis of $\Omega_{V | \mathbb{F}_q, P}^1 \otimes \mathcal{O}_{V_e, P}$. In particular, t_1, \dots, t_{m-e} all vanish on V and we may assume that they vanish even on Z , as by definition $V = Z \cap X$.

Let $\partial_1, \dots, \partial_n \in \mathcal{T}_{\mathbb{A}^n | \mathbb{F}_q, P}$ be the basis of the stalk of the tangent sheaf which is dual to dt_1, \dots, dt_n . We can find an $s \in A$ with $s(P) \neq 0$ such that $D_i = s\partial_i$ gives a global derivation $A \rightarrow A$ for $i = 1, \dots, n$. There exists a neighbourhood N_P of P in \mathbb{A}^n such that $N_P \cap U = N_P \cap \{t_{m+1} = \dots = t_n = 0\}$, $\Omega_{\mathbb{A}^n | \mathbb{F}_q}^1|_{N_P} = \bigoplus_{i=1}^n \mathcal{O}_{N_P} dt_i$ and $s \in \mathcal{O}(N_P)^*$. As U is quasi-compact, we can cover U with finitely many N_P and assume $U \subseteq N_P$. Hence in particular, $\Omega_{U | \mathbb{F}_q}^1 = \bigoplus_{i=1}^m \mathcal{O}_U dt_i$.

Let $P \in V_e$ be a closed point. For a polynomial $f \in I_d$, $H_f \cap U$ is not smooth at P if and only if $(D_1 f)(P) = \dots = (D_m f)(P) = 0$. Note that we do not have to require $f(P)$ to be zero, since Z is contained in the hypersurface H_f for $f \in I'_d$, and thus f vanishes at all points in $V_e \subseteq Z$ anyway.

Let $\tau = \max_{1 \leq i \leq l_e+1} (\deg t_i)$ and $\gamma = \lfloor (d - \tau)/p \rfloor$, where $l_e = \dim V_e$. We select $f_0 \in I'_d$ and $g_1 \in S'_\gamma, \dots, g_{l_e+1} \in S'_\gamma$ uniformly and independently at random. Then the distribution of

$$f = f_0 + g_1^p t_1 + \dots + g_{l_e+1}^p t_{l_e+1}$$

is uniform over I'_d . Note that by our assumption we have $e + l_e < m$ and since t_1, \dots, t_{m-e} vanish on Z , we get $t_1, \dots, t_{l_e+1} \in I'_d$.

Since the distribution of the polynomials f in this representation is uniform over I'_d , it is enough to bound the probability for an f constructed in this way to have a point $P \in V_{e, > \frac{d-c}{m+1}}$ such that $(D_1 f)(P) = \dots = (D_m f)(P) = 0$. Here we are using the construction above because by definition, the partial derivatives $D_i f = D_i f_0 + g_i^p s$ are independent of one another. We will select the polynomials $f_0, g_1, \dots, g_{l_e+1}$ one at a time.

For $0 \leq i \leq l_e + 1$, define

$$W_i = V_e \cap \{D_1 f = \dots = D_i f = 0\}.$$

Then $W_{l_e+1} \cap V_{e, > \frac{d-c}{m+1}}$ is the set of points $P \in V_e$ of degree $> \frac{d-c}{m+1}$ where $H_f \cap U$ may be singular. Using the induction argument of Lemma 2.6 of [Poo04], one can show that if the polynomials f_0, g_1, \dots, g_i for $0 \leq i \leq l_e$ have been chosen such that $\dim(W_i) \leq l_e - i$ holds, then the probability for $\dim(W_{i+1}) \leq l_e - i - 1$ is equal to $1 - o(1)$ as $d \rightarrow \infty$, and conditioned on a choice of f_0, g_1, \dots, g_{l_e} for which W_{l_e} is finite, the probability for $W_{l_e+1} \cap V_{e, > \frac{d-c}{m+1}}$ to be empty is equal to $1 - o(1)$ as $d \rightarrow \infty$. Hence, the probability for a polynomial f to have a point $P \in V_{e, > \frac{d-c}{m+1}}$

such that $H_f \cap U$ is not smooth at P is 0. The upper density $\overline{\mu}_Z(Q_V^{\text{high}})$, that we actually want to calculate, is a finite sum of those probabilities, and hence zero. \square

Proof of Theorem 1.1. By definition, $\mathcal{P} \subseteq \mathcal{P}_r \subseteq \mathcal{P} \cup Q_r^{\text{medium}} \cup Q_{U-V}^{\text{high}} \cup Q_V^{\text{high}}$. Thus $\overline{\mu}(\mathcal{P})$ and $\underline{\mu}(\mathcal{P})$ differ from $\mu(\mathcal{P}_r)$ at most by $\overline{\mu}_Z(Q_r^{\text{medium}}) + \overline{\mu}_Z(Q_{U-V}^{\text{high}}) + \overline{\mu}_Z(Q_V^{\text{high}})$. Using 3.2, 4.1 and 4.2 for singularities of medium and high degrees, we get

$$\mu_Z(\mathcal{P}) = \lim_{r \rightarrow \infty} \mu_Z(\mathcal{P}_r) = \frac{\#T}{\#H^0(Y, \mathcal{O}_Y)} \frac{\zeta_V(m+1)}{\zeta_U(m+1) \prod_{e=0}^{m-1} \zeta_{V_e}(m-e)}.$$

\square

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